Derivation of the Aerodynamic Forces for the Mesicopter Simulation

Gary Fay

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This document describes the derivation of the aerodynamic forces used in the equations of motion for stability and control analysis of the mesicopter.

1 Aerodynamic Forces

The derivation of the aerodynamic forces is resolved using a combination of momentum and blade element theory. Momentum theory is used to derive the inflow or induced velocity. The rotor is modeled as an actuator disk across which there is no discontinuity in velocity, but there is a discontinuity in pressure. The inflow velocity of the air when it crosses the disk, and is assumed to be uniform at all points on the disk. Using conservation of mass through the disk and conservation of energy in the wake leads to an equation for thrust of the form $T = 2\rho A \nu_1^2$. In hover, the thrust is equal to the weight so T is replaced by W. Solving for the inflow velocity in hover results in the following expression:

$$\nu_{1_{hov}} = \sqrt{\frac{W}{2\rho A}} \quad \text{[m/sec]} \ .$$

Now consider the case of the rotor moving sideways through the air. Momentum theory will produce a different expression for thrust, $T = 2\rho A \nu_1 \sqrt{V^2 + \nu_1^2}$. Note that the rotor disk is assumed to have zero angle of attack with respect to the incoming flow, i.e. the rotor shaft is perpendicular to the direction of travel. Solving for the inflow velocity in this case results in the final expression that will be used in all future calculations involving inflow:

$$\nu_1 = \sqrt{-\frac{V^2}{2} + \sqrt{\left(\frac{V^2}{2}\right)^2 + \left(\frac{W}{2\rho A}\right)^2}} \quad \text{[m/sec]} \ . \tag{1}$$

Note that for the case of arbitrary motion in three dimensions, the rotor velocity is most easily described in the body fixed shaft axes; Z-axis down parallel to the shaft, X-axis forward, and Y-axis out the right side of the vehicle. In this case, the sideways velocity of the rotor is $V = \sqrt{\dot{x}^2 + \dot{y}^2}$. Also, the total inflow velocity includes vertical motion of the rotor as well. The inflow ratio, λ, is a dimensionless quantity often used in helicopter literature to relate the inflow velocity to the rotor tip velocity. The inflow ratio is defined as:

$$\lambda = \frac{\nu_1 - \dot{z}}{\Omega R} \,.$$

Another dimensionless quantity used in helicopter literature is the rotor advance ratio, which relates the horizontal velocity with the tip velocity of a rotor. The definition of the rotor advance ratio is:

$$\mu = \frac{V}{\Omega R} \,,$$

where V is the horizontal velocity defined earlier.

Blade element theory is used to determine the total aerodynamic forces and torques acting on a rotor. As the name implies, these forces and torques are calculated by integrating the individual forces acting on smaller blade elements over the entire rotor. Figure 1 depicts a blade element as well as the velocities it sees and the forces acting on it. The "horizon" line is the line perpendicular to the rotor shaft, and will be parallel with the true horizon in hover. The geometric pitch or incidence, denoted by θ, is the angle between the horizon line and the blade chord line. The angle between the blade chord line and the local velocity vector is the local angle of attack, α, and the angle between the horizon and the local velocity vector is the local inflow angle, ϕ. The local velocity vector seen by each blade element is further decomposed into vertical and horizontal components, U_P and U_T. The vertical component of the local velocity vector is due to inflow and vertical motion of the rotor, so it is uniform for every section. The horizontal component is due to the angular velocity of

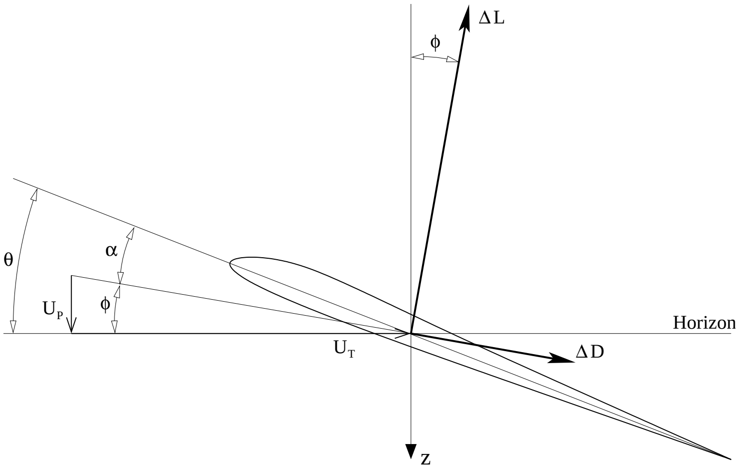

Figure 1: Blade Element including Local Velocities and Forces.

the blade element and horizontal motion of the rotor; therefore, it is depends on the radial position of the element as well as the current orientation of the blade with respect to the incoming horizontal flow. These quantities are defined as:

$$\begin{aligned} U_P &= \nu_1 - \dot{z} &&= \Omega R \lambda \,, \quad \text{and} && (2) \\ U_T &= \Omega r + V \sin \Psi &&= \Omega R \left(\frac{r}{R} + \mu \sin \Psi \right) \,, && (3) \end{aligned}$$

where Ψ is the azimuth angle. This is the angle the blade makes as it rotates about the shaft such that $\Psi = 0$ is defined to be in the direction of the horizontal velocity vector V, i.e. opposite the direction of motion of the vehicle. V was defined earlier as $V = \sqrt{\dot{x}^2 + \dot{y}^2}$.

The goal of blade element theory is to calculate the total forces acting parallel and perpendicular to the rotor shaft as well as the torques acting about the rotor shaft. The force parallel to the rotor shaft is defined as the thrust of the rotor, and the forces perpendicular to the rotor shaft are defined as the hub forces. In the following subsections, the derivation of the thrust, hub forces, and rotor torques will be described.

1.1 Thrust Derivation

Thrust is calculated by integrating the vertical forces acting on all the blade elements. In this case, vertical is defined as perpendicular to the horizon line. The lift force is perpendicular to the local velocity vector, while the drag force is parallel to the local velocity. Thus the vertical force acting on a blade section is $\Delta F_V = \Delta L \cos(\phi) - \Delta D \sin(\phi)$. Here, ΔL, ΔD, and ΔF_V correspond to the forces acting on a blade element section, Δr. The equations for lift and drag are obtained from any introductory aerodynamic text as:

$$\begin{aligned} \Delta L &= q C_l c \Delta r \,, \quad \text{and} \\ \Delta D &= q C_d c \Delta r \,. \end{aligned}$$

At this point many assumptions need to be made to obtain a closed form solution for the thrust. First, only aerodynamic lift and drag forces are assumed to act on a section. Forces out of plane with the section are dominated by the centrifugal force, and cancel if the rotor is balanced. Aerodynamic moments are assumed to be negligible by assuming the shear center and aerodynamic center of the blade lie close together, and that the blades are reasonably stiff. (In actuality, this may not be true for the smaller mesicopter blades.) The mesicopter blades are light enough so that gravity torques may be safely ignored. The blade is assumed to be rigid, so that there is no flapping or coning. Centrifugal stiffening also adds to this effect, and the entire rotor always lies in a plane. To define the reference area of the element, an assumption that the rotor blade has constant chord is made. For cases where the chord is not constant, average chord is used. The next assumption, which can be found in any introductory aerodynamics text, is that the coefficient of lift varies linearly with angle of attack by $C_l = a\alpha = a\left(\theta - \phi\right)$. Here a linear twist distribution is assumed so that the twist varies linearly with radial position according to the relation $\theta = \theta_0 - \theta_{tw}\,(r/R)$. Figure 2 shows a linear approximation of blade twist plotted on top of the actual twist distribution of the designed mesicopter blade. This approximation works well from one quarter of the radius out to the tip. The inboard twist is not as important since the lift produced in this region is small because the angular velocity, Ωr , approaches zero. The inflow angle, also needed to define the lift coefficient, may be approximated by assuming that the angular velocity of the blade section is much larger than the total inflow through the blade. Then small angle approximations can be used to define the inflow angle as $\phi = U_P/U_T$. One last assumption used later is that the lift acting on the blade is about an order of magnitude higher than the drag.

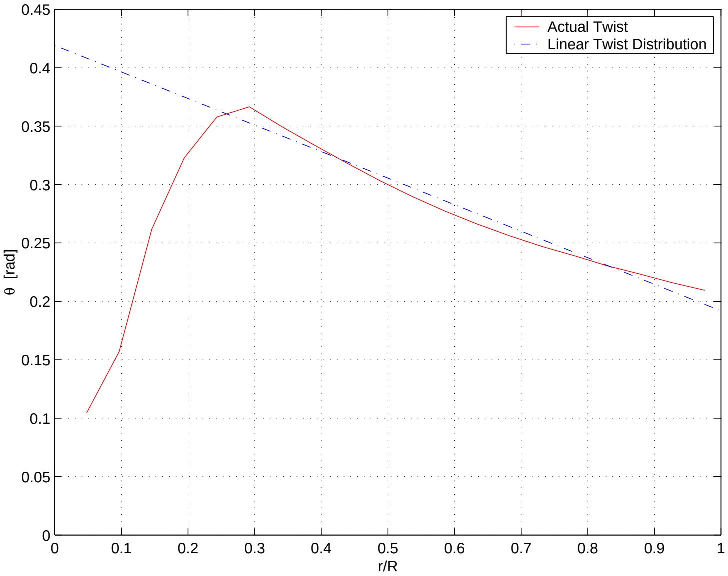

Figure 2: Comparison of Actual Blade Twist with Linear Twist Approximation.

The terms in the lift and drag equations can now be substituted with quantities that are well known. The dynamic pressure of the section is $q = \frac{1}{2}\rho U_T^2$. The reference area is defined using the average chord, $S = \bar{c}\Delta r$. And the coefficient of lift is replaced with the expression defined above resulting in the lift for a single blade element:

$$\Delta L = \frac{1}{2}\rho U_T^2 a \left(\theta_0 - \theta_{tw}\frac{r}{R} - \frac{U_P}{U_T}\right)\bar{c}\Delta r\,. \tag{4}$$

Similarly, an expression for the drag of the section can be developed:

$$\Delta D = \frac{1}{2}\rho U_T^2 \overline{C_d}\bar{c}\Delta r\,, \tag{5}$$

where $\overline{C_d}$ is the drag coefficient of the section corresponding to the 70% radial station. This may be assumed to be typical of the lift producing portion of the blade which is also the portion of the blade causing the majority of the drag.

The thrust is finally found by integrating the vertical forces acting on all the blade element sections. The vertical force was defined at the beginning of this section, but may be approximated by $\Delta F_V = \Delta L$. by applying small angle approximations for ϕ and using the fact that the thrust is an order of magnitude larger than the drag. Making use of the lift defined in Equation 4, and the velocity components given by Equations 2 and 3, the total thrust is computed:

$$\begin{aligned}
T &= \frac{N}{2\pi}\int_0^{2\pi}\int_0^R \frac{\Delta L}{\Delta r} dr d\Psi \\
&= \frac{N\rho a\bar{c}}{4\pi}\int_0^{2\pi}\int_0^R \left[U_T^2\left(\theta_0 - \theta_{tw}\frac{r}{R}\right) - U_T U_P\right] dr d\Psi \\
&= \frac{N\rho a\bar{c}\left(\Omega R\right)^2}{4\pi}\int_0^{2\pi}\int_0^R \left[\left(\frac{r}{R} + \mu\sin\Psi\right)^2\left(\theta_0 - \theta_{tw}\frac{r}{R}\right)\right. \\
&\quad \left. - \lambda\left(\frac{r}{R} + \mu\sin\Psi\right)\right] dr d\Psi \\
&= N\rho a\bar{c}\left(\Omega R\right)^2 R\left[\left(\frac{1}{6} + \frac{1}{4}\mu^2\right)\theta_0 - \left(1+\mu^2\right)\frac{\theta_{tw}}{8} - \frac{1}{4}\lambda\right].
\end{aligned}$$

It is common to use non-dimensional quantities when describing the aerodynamic forces. Aerodynamicists use the coefficient of thrust defined by:

$$C_T = \frac{T}{\rho A\left(\Omega R\right)^2}$$

Another important non-dimensional quantity used to simplify the thrust equation is the rotor solidity defined as the ratio of blade area to disk area. Using the average blade chord, then the solidity ratio is $\sigma = N\bar{c}/\pi R$. The coefficient of thrust can be computed by inserting the formula for thrust calculated above and simplifying:

$$\frac{C_T}{\sigma a} = \left(\frac{1}{6} + \frac{1}{4}\mu^2\right)\theta_0 - \left(1+\mu^2\right)\frac{\theta_{tw}}{8} - \frac{1}{4}\lambda. \qquad (6)$$

This is the relationship that will be used to derive the dynamics of the mesicopter.

1.2 Hub Force Derivation

The derivation of the hub force is nearly identical to the derivation of the thrust above, except that now the forces in the horizontal plane are integrated over all the blade elements. The hub force is actually decomposed into two orthogonal forces both perpendicular to the rotor shaft. The H-force lies along the

$\Psi = 0$ azimuth, and the Y-force lies along the azimuth $\Psi = \frac{\pi}{2}$. The horizontal force acting on a blade element is $\Delta F_H = \Delta D \cos(\phi) + \Delta L \sin(\phi)$. Small angle approximations can still be made, but now both terms are significant, so this expression only simplifies to $\Delta F_H = \Delta D + \Delta L \left(U_P / U_T \right)$. The lift and drag forces were defined in the previous section along with all the other terms that are needed to compute the hub force. First the H-force will be calculated by integrating only the component of the hub forces lying in the $\Psi = 0$ direction:

$$\begin{aligned}
H &= \frac{N}{2\pi} \int_0^{2\pi} \int_0^R \left[\frac{\Delta D}{\Delta r} + \frac{\Delta L}{\Delta r} \frac{U_P}{U_T} \right] \sin \Psi dr d\Psi \\
&= \frac{N \rho \bar{c}}{4\pi} \int_0^{2\pi} \int_0^R \left[U_T^2 \overline{C_d} + a \left(U_T U_P \left(\theta_0 - \theta_{tw} \frac{r}{R} \right) - U_P^2 \right) \right] \sin \Psi dr d\Psi \\
&= \frac{N \rho \bar{c} \left(\Omega R \right)^2}{4\pi} \int_0^{2\pi} \int_0^R \left[\left(\frac{r}{R} + \mu \sin \Psi \right)^2 \overline{C_d} \right. \\
&\quad \left. + a \left(\left(\frac{r}{R} + \mu \sin \Psi \right) \lambda \left(\theta_0 - \theta_{tw} \frac{r}{R} \right) - \lambda^2 \right) \right] \sin \Psi dr d\Psi \\
&= N \rho a \bar{c} \left(\Omega R \right)^2 R \left[\frac{1}{4a} \mu \overline{C_d} + \frac{1}{4} \lambda \mu \left(\theta_0 - \frac{\theta_{tw}}{2} \right) \right] .
\end{aligned}$$

Once again, a non-dimensional coefficient for the hub force can be defined as:

$$C_H = \frac{H}{\rho A \left(\Omega R \right)^2}$$

The final result that will be used when deriving the mesicopter dynamics is:

$$\frac{C_H}{\sigma a} = \frac{1}{4a} \mu \overline{C_d} + \frac{1}{4} \lambda \mu \left(\theta_0 - \frac{\theta_{tw}}{2} \right) . \tag{7}$$

The side force, Y, is similarly calculated by integrating the component of the hub forces lying in the $\Psi = \frac{\pi}{2}$ direction:

$$Y = -\frac{N}{2\pi} \int_0^{2\pi} \int_0^R \left[\frac{\Delta D}{\Delta r} + \frac{\Delta L}{\Delta r} \frac{U_P}{U_T} \right] \cos \Psi dr d\Psi .$$

Following the same steps as above results in the forces canceling in this direction, so the total side force is zero. Therefore the non-dimensional coefficient is also zero:

$$C_Y = 0 . \tag{8}$$

1.3 Torque Derivation

The aerodynamic forces acting on the blade elements also cause moments about the rotor shaft and hub. This section presents the derivations of the aerodynamic torques that act about the center of the rotor. In particular, the torque acting about the shaft is important because it determines the power required for the motor to keep the rotor spinning. The remaining torques cause the rotor to pitch or roll about the hub.

The rotor shaft and the vertical axis are colinear. Therefore, the torque about the rotor shaft is found by integrating the horizontal aerodynamic forces acting on each rotor section with a moment arm equal to the radius of that section. The horizontal forces acting on a rotor section have already been defined above as ΔF_H . In this case, the horizontal force is multiplied by the moment arm, r, and integrated over the entire rotor:

$$\begin{aligned} Q &= \frac{N}{2\pi}\int_0^{2\pi}\int_0^R\left[\frac{\Delta D}{\Delta r}+\frac{\Delta L}{\Delta r}\frac{U_P}{U_T}\right] r dr d\Psi \\ &= \frac{N\rho\bar{c}\left(\Omega R\right)^2}{4\pi}\int_0^{2\pi}\int_0^R\left[\left(\frac{r}{R}+\mu\sin\Psi\right)^2\overline{C_d}\right. \\ &\quad \left. +\,a\left(\left(\frac{r}{R}+\mu\sin\Psi\right)\lambda\left(\theta_0-\theta_{tw}\frac{r}{R}\right)-\lambda^2\right)\right] r dr d\Psi \\ &= N\rho a\bar{c}\left(\Omega R\right)^2 R^2\left[\frac{1}{8a}\left(1+\mu^2\right)\overline{C_d}+\lambda\left(\frac{1}{6}\theta_0-\frac{1}{8}\theta_{tw}-\frac{1}{4}\lambda\right)\right] . \end{aligned}$$

Using dimensionless quantities, the rotor torque coefficient is:

$$C_Q=\frac{Q}{\rho A\left(\Omega R\right)^2 R}$$

The final result is:

$$\frac{C_Q}{\sigma a}=\frac{1}{8a}\left(1+\mu^2\right)\overline{C_d}+\lambda\left(\frac{1}{6}\theta_0-\frac{1}{8}\theta_{tw}-\frac{1}{4}\lambda\right) . \tag{9}$$

Finally, the rolling and pitching moments will be considered. Because the blade will be moving horizontally through the air, the advancing blade sections will see larger horizontal velocity components than the retreating blade elements. The effect of this is that the advancing blade elements will produce more lift than the retreating blade elements resulting in an overall rolling moment being produced. The total rolling moment is calculated by integrating over the entire rotor the moments caused by the lift of each section acting at a moment arm equal to its radial position. The derivation follows:

$$\begin{aligned} R &= -\frac{N}{2\pi}\int_0^{2\pi}\int_0^R\frac{\Delta L}{\Delta r} r\sin\Psi dr d\Psi \\ &= -\frac{N\rho a\bar{c}\left(\Omega R\right)^2}{4\pi}\int_0^{2\pi}\int_0^R\left[\left(\frac{r}{R}+\mu\sin\Psi\right)^2\left(\theta_0-\theta_{tw}\frac{r}{R}\right)\right. \\ &\quad \left. -\,\lambda\left(\frac{r}{R}+\mu\sin\Psi\right)\right] r\sin\Psi dr d\Psi \\ &= -N\rho a\bar{c}\left(\Omega R\right)^2 R^2\mu\left(\frac{1}{6}\theta_0-\frac{1}{8}\theta_{tw}-\frac{1}{8}\lambda\right) . \end{aligned}$$

The non-dimensional rolling moment is found using the same method described for the rotor torque coefficient, C_Q. Take care not to confuse the rolling moment with the total blade radius. The context in which the symbol is used should make the definition obvious. The rolling torque coefficient is then defined as:

$$\frac{C_R}{\sigma a} = -\mu \left(\frac{1}{6}\theta_0 - \frac{1}{8}\theta_{tw} - \frac{1}{8}\lambda \right) . \tag{10}$$

The pitching moment calculation is developed in a similar manner except that the cosine of the azimuth angle is taken:

$$P \quad = \quad -\frac{N}{2\pi} \int_0^{2\pi} \int_0^{R} \frac{\Delta L}{\Delta r} r \cos \Psi dr d\Psi \, .$$

The result of this integration ends up being zero. Therefore the pitching torque coefficient is also zero:

$$C_P = 0 \, . \tag{11}$$